Derivation of the Airborne Speed Formula

\[ S = \frac{2.73 \times D}{\cos \theta \times \sqrt{h + D \times \tan \theta}} \]
Derivation of the Airborne Speed Formula

- This Derivation results in an equation that can be used for **ALL** airborne applications.
- To use the equation, you must determine three (3) things from the scene information:
  - The horizontal distance \( (D) \) traveled by c/m of the vehicle from the point of takeoff to **first touch**
  - The height \( (h) \) that the c/m is at when the vehicle reaches the \( (D) \) mentioned above (**first touch**)
  - AND, the takeoff angle \( \theta \) or the **percent of grade** \( (m) \) that exists at the point of launch.
**Derivation of the Airborne Speed Formula**

Measuring Vehicle Takeoff Angle

- $Y (-h)$
- $20^\circ$
- $180^\circ$
- $0(360)^\circ$
- $340^\circ$
- $270^\circ$

* Use Same System (360°) as you use in Momentum Analysis*
At that vault speed, I’ll never get them Dukes!
Derivation of the Airborne Speed Formula

\[ S = \frac{2.73 \times D}{\cos \theta \times \sqrt{h + D \times \tan \theta}} \]

- \( S \) = Speed at point of takeoff (MPH)
- \( D \) = Horizontal Distance in feet traveled by Car C/M
- \( h \) = Height C/M of Car from Point of Takeoff
- \( 2.73 \) = Mathematical Constant from Derivation
- \( \theta \) = Angle of Takeoff (slope) measured in degrees
Derivation of the Airborne Speed Formula

Equations are coupled w/time (Parametric)

We must look at the motion of the vehicle (speed) and first break it down into its horizontal (\(X\)) & vertical components (\(Y\))

Let's look at the horizontal (\(X\)) direction first

Step # 1

\[ X = Vo \cos \theta (T) \]
Derivation of the Airborne Speed Formula

Now once the vehicle goes airborne, for every second that it moves horizontally (X), it also is moving in some direction vertically (Y) as well.

Let's now look at the vertical (Y) direction.

+ y is in the same sense (direction) as gravity, t

\[ y = -V_0 \sin \theta(t) + \frac{1}{2} at^2 \]
Derivation of the Airborne Speed Formula

\[ D = V_0 \cos \theta (T) \]

X=The Horizontal Movement of the Car (Distance)
Therefore; \( X=D \)

Now \textit{re-write} first step and substitute \( D \) for \( X \)
Derivation of the Airborne Speed Formula

Now solve for $T$ in the Equation at Step #3

Step # 4

$$T = \frac{D}{Vo \cos \theta}$$

AND

$$T^2 = \frac{D^2}{Vo^2 \cos^2 \theta}$$
Derivation of the Airborne Speed Formula

Y = The Vertical Movement of the Car (Height)
Therefore; Y = h

So substitute h for Y in Step #2 and re-write

Step # 5

\[ h = -Vo \sin \theta(t) + \frac{1}{2} at^2 \]
Derivation of the Airborne Speed Formula

Remember at Step #4 we wrote equations for \( T \) and \( T^2 \)

So *re-write Step #5 and substitute the equations at Step #4 for \( T \) and \( T^2 \)*

**Step #6**

\[
h = -V_o \sin \theta \left( \frac{D}{V_o \cos \theta} \right) + \frac{1}{2} a \left[ \frac{D^2}{V_o^2 \cos^2 \theta} \right]
\]
Derivation of the Airborne Speed Formula

Take the Equation (Step #6) and replace a (acceleration) with g;

re-write

Step # 7

\[ h = -Vo \sin \theta \left( \frac{D}{Vo \cos \theta} \right) + \frac{1}{2} g \left[ \frac{D^2}{Vo^2 \cos^2 \theta} \right] \]

Step # 8

Now cancel Vo in the first term:

\[ h = -\sin \theta \left( \frac{D}{\cos \theta} \right) + \frac{1}{2} g \left[ \frac{D^2}{Vo^2 \cos^2 \theta} \right] \]
Derivation of the Airborne Speed Formula

Now take $\frac{1}{2}$ and $g$ in second term and multiply them both

Step # 9

$$h = -\sin \theta \left( \frac{D}{\cos \theta} \right) + \frac{g}{2} \left[ \frac{D^2}{V_0^2 \cos^2 \theta} \right]$$

Step # 10

Now multiply first two terms on left of equal sign

$$h = -\left( \frac{D \sin \theta}{\cos \theta} \right) + \frac{g}{2} \left[ \frac{D^2}{V_0^2 \cos^2 \theta} \right]$$
Derivation of the Airborne Speed Formula

Next take the $D$ and remove it from the fraction that was produced from last step. 

\[
D \times \frac{\sin \theta}{\cos \theta}
\]

**Rewrite:**

Step # 11

\[
h = -D \cdot \frac{\sin \theta}{\cos \theta} + \frac{g}{2} \left[ \frac{D^2}{V_0^2 \cos^2 \theta} \right]
\]

**SINCE:**

\[
\frac{\sin \theta}{\cos \theta} = \tan \theta
\]

**Substitute** terms and **rewrite:**

Gerry Murphy
Derivation of the Airborne Speed Formula

Step # 12

\[ h = -D \tan \theta + \frac{g}{2} \left[ \frac{D^2}{Vo^2 \cos^2 \theta} \right] \]

Now multiply terms on right of plus sign and rewrite:

\[ h = -D \tan \theta + \frac{g \cdot D^2}{2 \left[ Vo^2 \cos^2 \theta \right]} \]
Derivation of the Airborne Speed Formula

Next move the \textbf{minus term} across equal sign, \textit{this will} change its mathematical sign to positive; \textit{rewrite}:

\begin{equation}
\begin{aligned}
\ h + D \tan \theta &= \frac{g \times D^2}{2 \left[ V_0^2 \cos^2 \theta \right]}
\end{aligned}
\end{equation}

\textbf{REMEMBER}: The \textbf{purpose} of this exercise is to \textit{derive} an equation that determines \textit{speed (or Velocity)} of a vehicle that goes airborne. So we need to \textit{isolate that} \((V)\) in the equation
Derivation of the Airborne Speed Formula

Step # 14

\[
2 \left[ h + D \tan \theta \right] \cos^2 \theta \frac{g \cdot D^2}{g \cdot D^2} = \frac{1}{V_o^2}
\]

Next move both terms from Step 14 equation above across equal sign, this will invert each term and place Velocity term on proper side; rewrite:

Step # 15

\[
V_o^2 = \frac{g \cdot D^2}{2 \cos^2 \theta \left[ h + D \tan \theta \right]}
\]
Derivation of the Airborne Speed Formula

Next take the **square root of each term** in equation from Step 15. This will get rid of the squared terms, particularly, the Velocity squared on the left side of the equation.

**Step # 16**

\[ \sqrt{vo^2} = \frac{\sqrt{g} \ast \sqrt{D^2}}{\sqrt{2} \ast \sqrt{\cos^2 \theta} \ast \sqrt{[h + D \tan \theta]}} \]

Now, **Rewrite** the equation
Derivation of the Airborne Speed Formula

Step # 17

Derivation of the Airborne Speed Formula

\[ V_0 = \frac{\sqrt{g \times D}}{\sqrt{2 \cos \theta \sqrt{[h + D \tan \theta]}}} \]

Next take the square root of g (32.2) and divide that by the square root of 2.

\[ \frac{\sqrt{g}}{\sqrt{2}} = \frac{\sqrt{32.2}}{\sqrt{2}} = \frac{5.674}{1.414} = 4.01 \]
Derivation of the Airborne Speed Formula

Now *rewrite* the equation and *replace the square root of g in the upper term with 4.01* (square root of 2 has been divided out)

**Step # 18**

\[
 Vo = \frac{4.01 \cdot D}{\cos \theta \sqrt[ ]{h + D \tan \theta}}
\]

Velocity (Vo in this case) is equal to Speed multiplied by 1.466 (S x 1.466) So *rewrite* the equation and *replace* Vo with S x 1.466
Derivation of the Airborne Speed Formula

Step # 19

\[ S \times 1.466 = \frac{4.01D}{\cos \theta \sqrt{h + D \tan \theta}} \]

Next divide both sides of the equal sign by 1.466, this will leave S (Speed in mph) on the left side and 2.73 will replace 4.01 in the upper term on the right.

Now Rewrite the equation in its Final Form
Derivation of the Airborne Speed Formula

Step # 20

\[ S = \frac{2.73 \times D}{\cos \theta \times \sqrt{h + D \times \tan \theta}} \]
Summary

- This Derivation has yielded an equation that can be used for **ALL** airborne applications.
- The measurements needed for using the formula (Distance and height) are critical. The accuracy of your results are also very sensitive to this fact.
- Therefore a “Rule of Thumb” for its use in the field:
  - **Never** Measure D (Distance) to long or h (height) to short.
For Further Information & Practice

- Chapter VIII in the Textbook (FTAR)
- Readings on “Uniform Projectile Motion” in a College or Upper Level Physics Book
- Additional Exercises in the Textbook or re-working the Class Projects
THE END