Conservation of Linear Momentum & Occupant Kinematics

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CREATED FOR USE IN IPTM TRAINING PROGRAMS

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Conservation of Linear Momentum: COLM

- As Crash Reconstructionists, we have learned COLM can be a powerful tool for analysis.
- If we do a complete COLM analysis, we can find more than just impact speeds.
- We will explore how to use analysis to determine the magnitude and direction of the $\Delta v$ vectors.
- We will then see how to apply this information to occupant motion.
- We will first look at where COLM comes from and examine the boundaries on the analysis.

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Momentum Basics
Newton’s First Law of Motion

- A body at rest tends to remain at rest until acted upon by an external, unbalanced force. A body in motion tends to remain in motion until acted upon by external, unbalanced forces affecting that motion.
- Commonly called the law of **INERTIA**
Newton’s Second Law

- The acceleration of a body is directly proportional to the force acting on the body and inversely proportional to the mass of the body, in other words:
  - \( a = \frac{F}{m} \)
- More commonly written as: \( F=ma \)
Newton’s Third Law

- For every force acting on a body by another, there is an equal but opposite force reacting on the second body by the first.
- Sometimes stated: For every action there is an equal but opposite reaction.
- Equal & Opposite Forces
COLM 360° Coordinate System

FIGURE 1: The Standard 360° Left-Hand Coordinate System, pre-impact
Linear Momentum: Assumptions

- During a crash, all significant forces are between the vehicles.
- Wind resistance, tire forces acting during the very short time = negligible
  - These are external, impulsive forces
- So the two cars comprise a CLOSED system.
- Linear Momentum in a closed system doesn’t change so $M_{in} = M_{out}$
Linear Momentum

- Example: Two particles moving towards each other
Linear Momentum

- The two particles collide – The forces shown act on each particle

- Note: No external forces acting on either one
Linear Momentum

- The two particles now move away from each other

![Diagram showing two particles moving away from each other with arrows indicating their velocities.](image)
Linear Momentum

- Looking back to the collision itself....
Forces during a crash

- What forces act during a typical collision?
  - Vehicles acting on each other
  - Tire Forces
  - Aerodynamics
- Neglect small forces (tire, aero) because we can’t easily account for them, and they are << inter-vehicle forces
- Yields slightly conservative results
Newton’s Second Law says...

- Force = (mass) x (acceleration)

\[ F_1 = m_1 a_1 \]
\[ F_2 = m_2 a_2 \]
Newton’s Third Law says...

- Forces are equal and opposite

\[ F_1 = m_1 a_1 \]
\[ F_2 = m_2 a_2 \]

\[ F_1 = -F_2 \]
\[ m_1 a_1 = -m_2 a_2 \]
Definition of Acceleration

- acceleration = \frac{\text{Change in Velocity}}{\text{Change in Time}} = \frac{\Delta v_i}{\Delta t}

\[
F_1 = -F_2
\]

\[
m_1a_1 = -m_2a_2
\]

\[
m_1\left(\frac{\Delta v_1}{\Delta t}\right) = -m_2\left(\frac{\Delta v_2}{\Delta t}\right)
\]
Change in Linear Momentum

\[ F = ma = m\left(\frac{\Delta V}{\Delta t}\right) \]

\[ F(\Delta t) = m(\Delta V) \quad \text{Impulse} = \text{change in momentum} \]

So: \[ \Delta M = m(\Delta V) = F(\Delta t) \]

- Forces & times acting on each vehicle are the same, so \( \Delta M \) equal and opposite.
Conservation Of Linear Momentum: Linear Momentum IN = Linear Momentum OUT

\[ m_1 \Delta \nu_1 = -m_2 \Delta \nu_2 \]

\[ m_1 (\nu_3 - \nu_1) = -m_2 (\nu_4 - \nu_2) \]

\[ m_1 \nu_3 - m_1 \nu_1 = -m_2 \nu_4 + m_2 \nu_2 \]

\[ m_1 \nu_1 + m_2 \nu_2 = m_1 \nu_3 + m_2 \nu_4 \]
Elastic and Inelastic Collisions
Elastic and Inelastic Collisions

- We have derived a general COLM equation from Newton’s 2\textsuperscript{nd} and 3\textsuperscript{rd} Laws
- As long as the basic assumptions are met, these equations are valid for collision analysis.
- We need to understand the difference between elastic and inelastic collisions.
An elastic collision is simply one in which mechanical energy is conserved.

- Mechanical Energy is the sum of Kinetic and Potential Energies.
- Since collisions take place with no displacement, we may then define an elastic collision as one in which Kinetic Energy is conserved.
An inelastic collision is simply one in which Kinetic Energy is NOT conserved.

The work done to crush the vehicles is irreversible work.

- Essentially, this means the work, hence energy, used to crush the vehicles is transformed into other forms of energy, such as heat.
Elastic and Inelastic Collisions

- We consider traffic crashes at normal street and highway speeds to be inelastic.
- Our experience with controlled testing over the years tells us this is a reasonable assumption.
- However, some low-speed collisions will have some “bounce” to them.
- We describe this as the coefficient of restitution.
Coefficient of Restitution
Consider for a moment a collinear, central collision between two bodies.

Newton defined the coefficient of restitution as follows:

\[ \varepsilon = \frac{v_4 - v_3}{v_1 - v_2} \]
Coefficient of Restitution

Where:

\[ \varepsilon = \text{coefficient of restitution} \]
\[ v_4 = \text{post-impact velocity of body 2} \]
\[ v_3 = \text{post-impact velocity of body 1} \]
\[ v_1 = \text{impact velocity of body 1} \]
\[ v_1 = \text{impact velocity of body 2} \]

This is also known as the *kinematic* definition of restitution.
Using Coefficient of Restitution

We may apply the coefficient of restitution with the following equation:

\[
\nu_c = \frac{\Delta v_1 (m_1 + m_2)}{m_2 (\varepsilon + 1)}
\]
Using Coefficient of Restitution

Where:

\[ v_c = \text{closing velocity } (v_1 - v_2) \]
\[ \Delta v_1 = \text{Velocity change of body 1} \]
\[ m_1 = \text{mass of body 1} \]
\[ m_2 = \text{mass of body 2} \]
\[ \varepsilon = \text{coefficient of restitution} \]

This equation may be useful when \( \Delta v_1 \) is known, perhaps from an event data recorder.
Using Coefficient of Restitution

Some typical values for Coefficient of Restitution:
- $\Delta V$ above 15-20 mph: 0.0 to 0.15
- $\Delta V$ below 15 mph: 0.20 to 0.45

In low speed collisions, we will have to account for a coefficient of restitution.
If vehicles become entangled, then $\varepsilon = 0$
For a further discussion of the coefficient of restitution, see:


and, for a more in-depth discussion:

Dealing With External Forces
What about external forces?

Is Linear Momentum Conserved?
Free Body Diagram

\[ F_{\text{TREE}} = m_1 a_1 + F_{\text{EXT}} \]
\[ F_{\text{CAR}} = m_2 a_2 \]
Equations Including Significant External Force on Body 1

- **Third Law:** \( F_{\text{TREE}} = -F_{\text{CAR}} \)
  
  \[
  m_1 a_1 + F_{\text{EXT}} = -m_2 a_2
  \]

- **Definition of acceleration:**
  
  \[
  m_1 \left( \frac{\Delta v_1}{\Delta t} \right) + F_{\text{EXT}} = -m_2 \left( \frac{\Delta v_2}{\Delta t} \right)
  \]

- **Multiply through by Delta t:**
  
  \[
  m_1 \Delta v_1 + F_{\text{EXT}} \Delta t = -m_2 \Delta v_2
  \]
Linear Momentum Equation with Impulse

\[ m_1 \Delta v_1 + F_{EXT} \Delta t = -m_2 \Delta v_2 \]
\[ m_1 (v_3 - v_1) + F_{EXT} \Delta t = -m_2 (v_4 - v_2) \]
\[ m_1 v_3 - m_1 v_1 + F_{EXT} \Delta t = -m_2 v_4 + m_2 v_2 \]

- **Rearrange:**

  \[ m_1 v_1 + m_2 v_2 = m_1 v_3 + m_2 v_4 + F_{ext} \Delta t \]

  - Momentum In
  - Momentum Out
  - External Impulse
Is Linear Momentum Conserved?

- No, the presence of external forces are significant.
- The system is no longer closed.
Is Momentum Conserved?

- Tree is stationary and does not have measurable momentum (in or out)
- Newton’s Laws still work
- Large external forces over a small time can influence momentum solutions
- Tire forces over 0.1 seconds are typically negligible for similar vehicles except for low speed collisions
What's Wrong With This Analysis?

The large external, impulsive force from the pole is ignored.

Even though linear momentum may not be conserved, the occupants will still move toward the applied force!

Image from: "Momentum Equations - How they work for the reconstructionist" Westmoreland, TEEX

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Collision Types
Collision Types

Collinear Collisions
• **Collinear collisions**
  - Also called “inline collisions.”
  - Where the approach velocity vectors are parallel.
  - Collision can be “central” or “non-central.”
  - All movement is one dimensional, along the $x$-axis.
Collision Types (cont.)

- **Collinear collisions** (cont).
  - Momentum is a vector quantity so the direction of the velocity vectors (both approach and departure) must be taken into account.
  - Movement to the right is towards $0^\circ$ (the positive $x$-axis) so the movement carries a positive sign.
  - Movement to the left is towards $180^\circ$ (the negative $x$-axis) so the movement carries a negative sign.
  - If the appropriate sign is not used, a huge error in calculated speeds can result!
Collinear collisions (cont).

- Some formula books do not list enough formulas that cover all approach and departure scenarios.
- Thus, the unsuspecting investigator, using the formula with incorrect signs can have a huge error in the calculated speed.
- Using direction cosines in the equation will alleviate this problem.
Collision Types (cont.)

- Collinear collisions (cont).
  - General momentum equation with direction cosines:
    \[ \delta_1 w_1 v_1 + \delta_2 w_2 v_2 = \delta_3 w_1 v_3 + \delta_4 w_2 v_4 \]
  - The \( \delta \) symbol is the lower case Greek letter delta.
  - The value of \( \delta \) will either be +1 or -1 depending on the direction of the velocity vector.
Collinear collisions (cont).

- If there is a small angle (10° or less) between the approach vectors, relative to each other, the collision may lend itself to a collinear collision solution using only direction cosines.

- However, depending on the particular crash, a damage-momentum solution or a simultaneous equation solution may be more appropriate.
Collision Types

Two Dimensional Collisions
Two dimensional collisions

- In this type of collision, the approach velocity vectors of the vehicles are not parallel and make some angle with respect to each other.
- This type of collision is also called a *planar* collision.
- The collision can be central or non-central.
- A two dimensional momentum analysis is generally performed to calculate approach or impact speeds.
Collision Types (cont.)

- **Two dimensional collisions** (cont).
  - Older terminology often referred to this type of collision as an angular collision.
  - However, older terminology also referred to the momentum solution of this collision as an “angular momentum” analysis.
  - **THIS IS INCORRECT!**
  - Two dimensional collisions are **LINEAR momentum problems**, NOT angular.
Two dimensional collisions (cont).

- Angular momentum is something that a spinning or rotating object possesses.
- The formula for angular momentum is:

\[ Q = I \omega \]

Where

- \( Q \) is the angular momentum
- \( I \) is the moment of inertia
- \( \omega \) is the angular velocity.
Collision Types (cont.)

- Two dimensional collisions (cont).
  - Recall the general momentum equation:

\[ \mathbf{w}_1 \mathbf{v}_1 + \mathbf{w}_2 \mathbf{v}_2 = \mathbf{w}_1 \mathbf{v}_3 + \mathbf{w}_2 \mathbf{v}_4 \]

  - Momentum is a vector, possessing both magnitude and direction.
  - In a collinear collision, the directional components of the momentum vectors were taken care of by the direction cosines.
  - In a two dimensional momentum solution, the directional component of the momentum vectors still must be addressed.
• Two dimensional collisions (cont).
  The directional components can be taken care of by introducing sine and cosine elements to the general momentum equation:

  $$ x - \text{direction} $$

  \[ w_1v_1 \cos \alpha + w_2v_2 \cos \psi = w_1v_3 \cos \theta + w_2v_4 \cos \phi \]

  $$ y - \text{direction} $$

  \[ w_1v_1 \sin \alpha + w_2v_2 \sin \psi = w_1v_3 \sin \theta + w_2v_4 \sin \phi \]
Two dimensional collisions (cont).

Solving both equations for the approach or impact speeds, \( v_1 \) and \( v_2 \) yields:

\[
v_2 = \frac{w_1 v_3 \sin \theta}{w_2 \sin \psi} + \frac{v_4 \sin \varphi}{\sin \psi}
\]

\[
v_1 = v_3 \cos \theta + \frac{w_2 v_4 \cos \phi}{w_1} - \frac{w_2 v_2 \cos \psi}{w_1}
\]
Determining Post-Impact Directions
• **Impact circle**
  - The concept of the *impact circle* was first presented by Dr. Gordon Bigg during his presentation *Momentum – Facts and Myths* at the 1998 IPTM Special Problems conference.
  - The impact circle is a region in space and time where collision forces act upon the vehicles.
  - A COLM analysis doesn’t consider what is happening to the vehicles during the collision phase.
  - The impact circle can be thought of as a cloud that covers and obscures the area of impact.
Determining Post-Impact Direction (cont.)

- Impact circle (cont).
Determining Post-Impact Direction (cont.)

- **Impact circle (cont).**
  - The impact circle includes secondary slaps.
  - Secondary slaps do not have to be treated separately.
  - The principle of conservation of linear momentum states that the \textit{total} momentum before a collision is equal to the \textit{total} momentum after the collision.
  - Momentum is exchanged during the collision.
  - Nothing says that this exchange has to occur in one impact or two.
Impact circle (cont).
- Departure angles are obtained by determining the direction the velocity vectors of the vehicles are heading at the point where the vehicles have stopped interacting (collision forces).
- The directions (angles) are measured with respect to the x-axis.
- Departure angles are **NOT** measured from impact to final position.
- Departure angles also are not measured from first contact to separation.
COLM - Vector Analysis
Putting it Together: Vector Analysis

- We will use the following conventions for our variables:

  - $S_1 =$ Vehicle 1 Impact speed
  - $S_2 =$ Vehicle 2 Impact speed
  - $S_3 =$ Vehicle 1 Post-impact speed
  - $S_4 =$ Vehicle 2 Post-impact speed
  - $\Psi =$ Approach angle of Vehicle 2
  - $\theta =$ Departure angle of Vehicle 1
  - $\phi =$ Departure angle of Vehicle 2
  - $w_1 =$ weight of vehicle 1 in lb
  - $w_2 =$ weight of vehicle 2 in lb
  - $\Delta S_1 =$ Velocity change of vehicle 1
  - $\Delta S_2 =$ Velocity change of vehicle 2
  - $\alpha_1 =$ PDOF angle vehicle 1
  - $\alpha_2 =$ PDOF angle vehicle 2

The approach angle of vehicle 1 is always 0° or 180°.
Example:

Unit #1 is traveling eastbound on Main St., Unit #2 is traveling northbound on Ash St. Both units collide in the intersection at a right angle with Unit #1 departing the collision at an angle of 40° and Unit #2 departing the collision at an angle of 25°. Unit #1’s departure speed was 30 mph and Unit #2’s departure speed was 20 mph.

Unit #1 weighs 3000 lbs. And Unit #2 weighs 2000 lbs.

Determine the impact speeds \(v_1\) and \(v_2\).
## MATHEMATICAL SOLUTION

<table>
<thead>
<tr>
<th>Set up the data table</th>
<th>VEHICLE 1</th>
<th>VEHICLE 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Weight</strong></td>
<td>$W_1 = 3,000$ lb</td>
<td>$W_2 = 2,000$ lb</td>
</tr>
<tr>
<td><strong>Approach Speed</strong></td>
<td>$S_1 = ____$ mph</td>
<td>$S_2 = ____$ mph</td>
</tr>
<tr>
<td><strong>Approach Angle</strong></td>
<td>$\alpha = 0$</td>
<td>$\psi = 90$</td>
</tr>
<tr>
<td></td>
<td>$\cos \alpha = 1.000$</td>
<td>$\cos \psi = 0.0000$</td>
</tr>
<tr>
<td></td>
<td>$\sin \alpha = 0.000$</td>
<td>$\sin \psi = 1.000$</td>
</tr>
<tr>
<td><strong>Departure Speed</strong></td>
<td>$S_3 = 30$ mph</td>
<td>$S_4 = 20$ mph</td>
</tr>
<tr>
<td><strong>Departure Angle</strong></td>
<td>$\theta = 40$ degrees</td>
<td>$\phi = 25$ degrees</td>
</tr>
<tr>
<td></td>
<td>$\cos \theta = 0.7660$</td>
<td>$\cos \phi = 0.9063$</td>
</tr>
<tr>
<td></td>
<td>$\sin \theta = 0.6428$</td>
<td>$\sin \phi = 0.4226$</td>
</tr>
</tbody>
</table>

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The Workhorse Equations

\[ S_2 = \frac{w_1 S_3 \sin \theta}{w_2 \sin \psi} + \frac{S_4 \sin \phi}{\sin \psi} \]

\[ S_1 = S_3 \cos \theta + \frac{w_2 S_4 \cos \phi}{w_1} - \frac{w_2 S_2 \cos \psi}{w_1} \]

(Veh.1 pre-crash direction = 0 degrees)
The Workhorse Equations: Solve for $S_2$ first

\[ S_2 = \frac{w_1 S_3 \sin \theta}{w_2 \sin \psi} + \frac{S_4 \sin \phi}{\sin \psi} \]

Do the Math:

\[ S_2 = 37.33 \text{ mph} \]

NOTE: Veh.1 pre-crash direction = 0 degrees
The Workhorse Equations: Solve for $S_1$ next

$$S_1 = S_3 \cos \theta + \frac{w_2 S_4 \cos \phi}{w_1} - \frac{w_2 S_2 \cos \psi}{w_1}$$

Do the Math:

$$S_1 = 35.06 \text{ mph}$$

NOTE: Veh.1 pre-crash direction = 0 degrees
Compare these speeds to the calculated $S_1$ & $S_2$ !!!!!

(35.0 mph and 37.3 mph respectively)
\[ \Delta S_1 = \sqrt{S_1^2 + S_3^2 - 2S_1S_3 \cos \theta} \]

\[ \Delta S_2 = \sqrt{S_2^2 + S_4^2 - 2S_2S_4 \cos(\psi - \phi)} \]
Calculate $\Delta V$ ($\Delta S$)

\[
\Delta S_1 = \sqrt{S_1^2 + S_3^2 - 2S_1 S_3 \cos \theta}
\]
\[
\Delta S_2 = \sqrt{S_2^2 + S_4^2 - 2S_2 S_4 \cos(\psi - \phi)}
\]

- $S_1 = 35.06$ mph
- $S_3 = 30.00$ mph
- $\Theta = 40^\circ$
- Do the Math:
  - $\Delta S_1 = 22.75$ mph

- $S_2 = 37.33$ mph
- $S_4 = 20.00$ mph
- $(\Psi - \Phi) = (90 - 25)^\circ = 65^\circ$
- Do the Math:
  - $\Delta S_2 = 34.10$ mph

- Potentially Fatal for UNRESTRAINED occupants of Unit #2!
Plot the change in momentum vectors:

Draw a line from the head of the $M_1$ vector to the head of the $M_3$ vector.

Put an arrowhead on this line at the $M_3$ end.

This is the $\Delta M_1$ (change in momentum) vector for Unit #1.

Label it $\Delta M_1$. 
Plot the change in momentum vectors:

Draw a line from the head of the $M_2$ vector to the head of the $M_4$ vector.

Put an arrowhead on this line at the $M_4$ end.

This is the $\Delta M_2$ (change in momentum) vector for Unit #2.

Label it $\Delta M_2$. 
These change in momentum vectors are also the:

**IMPULSE VECTORS!**

Per Newton’s Third Law they should be equal and opposite. The lengths should be equal and they should be parallel.

**CHECK IT !!!**
Calculate each vehicle’s $\Delta v$.

Unit #1

$\Delta M_1 = W_1 \Delta S_1$

$\Delta S_1 = \frac{M_1}{W_1}$

= 22.66 mph

Unit #2

$\Delta M_2 = W_2 \Delta S_2$

$\Delta S_2 = \frac{M_2}{W_2}$

= 34.00 mph

Potentially Fatal for UNRESTRAINED occupants of Unit #2.
\[ \alpha_1 = \sin^{-1} \left[ \frac{S_3 \sin \theta}{\Delta S_1} \right] \]

\[ \alpha_2 = \sin^{-1} \left[ \frac{S_4 \sin(\psi - \varphi)}{\Delta S_2} \right] \]
Calculate PDOF Angles

\[
\alpha_1 = \sin^{-1} \left[ \frac{S_3 \sin \theta}{\Delta S_1} \right]
\]

\[
\alpha_2 = \sin^{-1} \left[ \frac{S_4 \sin(\psi - \varphi)}{\Delta S_2} \right]
\]

- \( S_3 = 30.00 \text{ mph} \)
- \( \Delta S_1 = 22.75 \text{ mph} \)
- \( \Theta = 40^\circ \)
- Do the Math:
  - \( \alpha_1 = 57.95^\circ \)

- \( S_4 = 20.00 \text{ mph} \)
- \( \Delta S_2 = 34.10 \text{ mph} \)
- \( (\psi - \varphi) = 65^\circ \)
- Do the Math:
  - \( \alpha_2 = 32.11^\circ \)
PDOF Convention
Determine PDOF angles.

Extend the tail of $\Delta M_1$ past the x-axis using a dashed line. The angle this dashed line makes with the x-axis is the PDOF angle for Unit #1. Label it $\alpha_1$. Measure $\alpha_1$: $(58^0)$ (Calculated Value: $57.95^\circ$)
Determine PDOF angles.

\[ \alpha_2 = -32^0 \]

Extend the tail of \( \Delta M_2 \) past the y-axis using a dashed line. The angle this dashed line makes with the y-axis is the PDOF angle for Unit #2. Label it \( \alpha_2 \).

Measure \( \alpha_2 \). \((-32^0)\) (Calculated value \(-32.11^0\))
PDOF Parallel Check

- $\Psi = 180^\circ - PDOF_1 - PDOF_2$
  - OR
- $\Psi = 180^\circ - \alpha_1 - \alpha_2$
COLM – Multiple Departures
Recall the basic definition of COLM:

\[ w_1 v_1 + w_2 v_2 = w_1 v_3 + w_2 v_4 \]

Momentum in = Momentum out

- Momentum brought to the collision by Unit #1
- Momentum brought to the collision by Unit #2
- Exchange of Momentum.
- Momentum leaving the collision with Unit #1
- Momentum leaving the collision with Unit #2
• Add in the directional components to the momentum vectors:

\[ w_1 v_1 \cos \alpha + w_2 v_2 \cos \psi = w_1 v_3 \cos \theta + w_2 v_4 \cos \varphi \]

\textit{x - direction}

\[ w_1 v_1 \sin \alpha + w_2 v_2 \sin \psi = w_1 v_3 \sin \theta + w_2 v_4 \sin \varphi \]

\textit{y - direction}
Solve for $v_2$ and $v_1$:

$$v_2 = \frac{w_1 v_3 \sin \theta}{w_2 \sin \psi} + \frac{v_4 \sin \phi}{\sin \psi}$$

$$v_1 = v_3 \cos \theta + \frac{w_2 v_4 \cos \phi}{w_1} - \frac{w_2 v_2 \cos \psi}{w_1}$$
The equations just seen were for a “standard” two dimensional collision where there were two units in and two units out.

However, what happens if one or more units breaks apart during the collision or there is a separation of a load from the vehicle carrying it?

The departure portion of the momentum equations (right side of the equations) must be modified to account for all the significant pieces departing the collision.
The Crash
During the 2008 IPTM Special Problems in Traffic Crash Reconstruction conference two crash tests were conducted which involved two units in and four units out.

The first crash test was between a 1999 Chevrolet Cavalier bullet vehicle and a 1999 Pontiac Grand Am target vehicle towing a jet ski on a light trailer.
Vehicles

- 1999 Chevrolet Cavalier LS
• 1999 Pontiac Grand Am SE
Vehicles (cont.)

- Sea Doo 951XP
- Continental Trailer
2008 IPTM Special Problems
First Crash Test – Friday April 18, 2008
Analysis
Equations (cont.)

- Modified \( v_2 \) momentum equation for two in – multiple out

\[
v_2 = \frac{w_1 v_3 \sin \theta}{w_2 \sin \psi} + \frac{v_4 \sin \varphi}{\sin \psi}
\]

\[
v_2 = \frac{w_1 v_3 \sin \theta}{w_2 \sin \psi} + \frac{w_{GA} v_{GA} \sin \phi}{w_2 \sin \psi} + \frac{w_{js} v_{js} \sin \gamma}{w_2 \sin \psi} + \frac{w_{trl} v_{trl} \sin \beta}{w_2 \sin \psi}
\]
Modified \( v_1 \) momentum equation for two in – multiple out

\[
v_1 = v_3 \cos \theta + \frac{w_2 v_4 \cos \phi}{w_1} - \frac{w_2 v_2 \cos \psi}{w_1}
\]

\[
v_1 = v_3 \cos \theta + \frac{w_{GA} v_{GA} \cos \phi}{w_1} + \frac{w_{js} v_{js} \cos \gamma}{w_1} + \frac{w_{trl} v_{trl} \cos \beta}{w_1} - \frac{w_2 v_2 \cos \psi}{w_1}
\]
Equations (cont.)

- However....
  - Since the jet ski and the trailer had the same departure angle and speed, we can consolidate the equations:

\[
\begin{align*}
  v_2 &= \frac{w_1 v_3 \sin \theta}{w_2 \sin \psi} + \frac{w_{GA} v_{GA} \sin \phi}{w_2 \sin \psi} + \frac{w_{jstrl} v_{jstrl} \sin \gamma}{w_2 \sin \psi} \\
  v_1 &= v_3 \cos \theta + \frac{w_{GA} v_{GA} \cos \phi}{w_1} + \frac{w_{jstrl} v_{jstrl} \cos \gamma}{w_1} - \frac{w_2 v_2 \cos \psi}{w_1}
\end{align*}
\]
### Data Table: Unit #1 Cavalier, Unit #2 Grand Am, jet ski & trailer, Grand Am, Jet ski, Trailer

<table>
<thead>
<tr>
<th></th>
<th>Unit #1 Cavalier</th>
<th>Unit #2 Grand Am, jet ski &amp; trailer</th>
<th>Grand Am</th>
<th>Jet ski</th>
<th>Trailer</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Weight</strong></td>
<td>2617 lb</td>
<td>4131 lb</td>
<td>3116 lb</td>
<td>815 lb</td>
<td>200 lb</td>
</tr>
<tr>
<td><strong>Approach Speed</strong></td>
<td>$S_1$</td>
<td>$S_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Approach Angle</strong></td>
<td>$\alpha = 0^\circ$</td>
<td>$\psi = 121.5^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sin $\alpha$</td>
<td>0</td>
<td>Sin $\psi = 0.852$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cos $\alpha$</td>
<td>1</td>
<td>Cos $\psi = -0.522$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Departure Speed</strong></td>
<td>26.57 mph</td>
<td></td>
<td>21.17 mph</td>
<td>26.57 mph</td>
<td>26.57 mph</td>
</tr>
<tr>
<td><strong>Departure Angle</strong></td>
<td>$\theta = 15^\circ$</td>
<td>$\varphi = 119.5^\circ$</td>
<td>$\gamma = 15^\circ$</td>
<td>$\beta = 15^\circ$</td>
<td></td>
</tr>
<tr>
<td>Sin $\theta$</td>
<td>0.258</td>
<td>Sin $\varphi = 0.870$</td>
<td>Sin $\gamma = 0.258$</td>
<td>Sin $\beta = 0.258$</td>
<td></td>
</tr>
<tr>
<td>Cos $\theta$</td>
<td>0.965</td>
<td>Cos $\varphi = -0.492$</td>
<td>Cos $\gamma = 0.965$</td>
<td>Cos $\beta = 0.965$</td>
<td></td>
</tr>
</tbody>
</table>
## Results

### Data Table:

<table>
<thead>
<tr>
<th></th>
<th>Unit #1 Cavalier</th>
<th>Unit #2 Grand Am, jet ski &amp; trailer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>2617 lb</td>
<td>4131 lb</td>
</tr>
<tr>
<td>Approach Speed</td>
<td>42.42 mph</td>
<td>23.36 mph</td>
</tr>
<tr>
<td>Delta V</td>
<td>18.16 mph</td>
<td>2.40 mph</td>
</tr>
<tr>
<td>Speed at the start of skid</td>
<td>44.64 mph</td>
<td></td>
</tr>
</tbody>
</table>
Stalker Radar

Max speed 44.67 mph

SPEED, mph

TIME, sec

STALKER ATS

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System Status At Deployment

<table>
<thead>
<tr>
<th>Status</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shift Warning Lamp Status</td>
<td>OFF</td>
</tr>
<tr>
<td>Driver's Seat Belt Switch Status</td>
<td>UNBUCKLED</td>
</tr>
<tr>
<td>Passenger Front Air Bag Status</td>
<td>Air Bag Not Suppressed</td>
</tr>
<tr>
<td>Ignition Cycles At Deployment</td>
<td>21844</td>
</tr>
<tr>
<td>Ignition Cycles At Investigation</td>
<td>21845</td>
</tr>
<tr>
<td>Time From Algorithm Enable To Deployment Command (msec)</td>
<td>20</td>
</tr>
<tr>
<td>Time Between Non-Deployment And Deployment Events (sec)</td>
<td>N/A</td>
</tr>
</tbody>
</table>

![Graph showing recorded velocity change over time](image)

Max delta-V -14.70 mph

<table>
<thead>
<tr>
<th>Time (milliseconds)</th>
<th>Velocity change (MPH)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>117</td>
<td>-1.66</td>
</tr>
<tr>
<td>234</td>
<td>-2.19</td>
</tr>
<tr>
<td>351</td>
<td>-2.85</td>
</tr>
<tr>
<td>468</td>
<td>-3.51</td>
</tr>
<tr>
<td>585</td>
<td>-3.73</td>
</tr>
<tr>
<td>702</td>
<td>-3.95</td>
</tr>
<tr>
<td>819</td>
<td>-4.17</td>
</tr>
<tr>
<td>936</td>
<td>-5.05</td>
</tr>
<tr>
<td>1053</td>
<td>-5.92</td>
</tr>
<tr>
<td>1170</td>
<td>-7.24</td>
</tr>
</tbody>
</table>

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“To COLM or not to COLM...”

Situations where a Conservation of Linear Momentum analysis may not be the best choice...
High Mass / Momentum Ratio Collisions
In the photo and video just presented, we see there is a large mass ratio between the two colliding vehicles. The smaller vehicles were stopped, while the larger vehicles possessed all of the system momentum. In cases where either large mass or momentum ratios exist, we can say a lot about the speed of the vehicle possessing the larger momentum. However, it may be problematic to determine the speed of the vehicle with lesser momentum.
Let’s consider the following collision:

- Vehicle 1 is a 1992 White-GMC WG42T Tractor pulling a 2000 Fontaine drop-deck semi-trailer with 6000 lb of cargo. The total combination weight is 39,600 lb.
- Vehicle 2 is a 1994 Chevrolet S-10 Blazer weighing 3450 lb.
- The mass ratio between the two is about 11.5:1

The Blazer was stopped and the White TT impacted it in the passenger side with a known impact speed of 35.9 mph.

The two moved off as one unit at a speed of 32 mph.
We will first analyze this collision with the closing velocity equation:

\[
v_c = \frac{\Delta v_1 (m_1 + m_2)}{m_2 (\varepsilon + 1)}
\]

\[
= \frac{32(39,600 + 3450)}{39,600(0+1)}
\]

\[
= 34.78 \text{ mph}
\]
What if we were working this on the street and thought we had a 270° collision with a post-impact direction of 356°?

This post-impact direction may be present even if the Blazer was stopped because of post-impact steering forces from the TT unit.

We will examine how this potentially two-dimensional problem will effect our impact speeds.
### High Mass / Momentum Ratio Collisions

<table>
<thead>
<tr>
<th>Data Table:</th>
<th>Vehicle 1</th>
<th>Vehicle 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>39,600 lb</td>
<td>3450 lb</td>
</tr>
<tr>
<td>Approach Speed</td>
<td>$S_1$</td>
<td>$S_2$</td>
</tr>
<tr>
<td>Approach Angle</td>
<td>$\alpha = 0^\circ$</td>
<td>$\psi = 270^\circ$</td>
</tr>
<tr>
<td></td>
<td>Sin $\alpha = 0$</td>
<td>Sin $\psi = -1$</td>
</tr>
<tr>
<td></td>
<td>Cos $\alpha = 1$</td>
<td>Cos $\psi = 0$</td>
</tr>
<tr>
<td>Departure Speed</td>
<td>32 mph</td>
<td>32 mph</td>
</tr>
<tr>
<td>Departure Angle</td>
<td>$\theta = 356^\circ$</td>
<td>$\varphi = 356^\circ$</td>
</tr>
<tr>
<td></td>
<td>Sin $\theta = 0.070$</td>
<td>Sin $\varphi = 0.070$</td>
</tr>
<tr>
<td></td>
<td>Cos $\theta = 0.998$</td>
<td>Cos $\varphi = 0.998$</td>
</tr>
</tbody>
</table>
\[ S_2 = \frac{w_1 S_3 \sin \theta}{w_2 \sin \psi} + \frac{S_4 \sin \phi}{\sin \psi} \]

\[ = \frac{39,600(32)(-0.070)}{3450(-1)} + \frac{32(-0.070)}{-1} \]

\[ = 25.71 + 2.24 \]

\[ = 27.95 \text{ mph} \]
\[ S_1 = S_3 \cos \theta + \frac{w_2 S_4 \cos \phi}{w_1} - \frac{w_2 S_2 \cos \psi}{w_1} \]

\[ = 32(0.998) + \frac{3450(32)(0.998)}{39,600} - \frac{3450(32)(0)}{39,600} \]

\[ = 31.94 + 2.79 \]

\[ = 34.73 \text{mph} \]
In the preceding analysis, the mass ratio between the two vehicles was 11.5:1.

In the actual crash test, the Blazer was stopped, and the actual impact speed of the TT was 35.9 mph.

When we analyzed the problem with a momentum analysis assuming the Blazer was stopped, we calculate an impact speed for the TT of 34.78 mph.
One issue we have not yet addressed with regard to high mass ratio collisions is the presence of ground frictional forces generated by the larger vehicle. For example, an 80,000 lb TT unit with its parking brake applied may be able to generate a ground frictional force of 40,000 lb. A small vehicle impacting this TT unit may not even cause it to move if it were stopped, yet there could be extensive damage to the smaller vehicle.
Under-ride Collisions

- Passenger vehicles can hit and under-ride the semi-trailer of a TT unit.
- Sometimes, the TT unit, if stopped, will be moved ahead by a measurable distance.
- It is tempting to use a COLM analysis to determine the impact speed of the passenger vehicle.
- As we will see from the following crash test, this may be problematic.
Under-ride Collisions, cont’d.

- As we see in the video, the Jeep is forced down hard into the pavement.
- This provides a significant ground impulse that is difficult to quantify.
- In addition, we must also quantify the ground frictional impulse from the TT unit.
  - While this is something we can do, there will be some variance in the data that will require careful measurement.
- A COLM solution gave an impact speed range for the Jeep as 38 to 49 mph. Its actual impact speed was 37 mph.
If two similar vehicles collide in a near head-on configuration, it may be tempting to use a planar momentum analysis to determine impact speeds. However, even though linear momentum may be conserved in this collision, the analysis may not be useful. As we will see, for shallow approach angles, the sine value changes rapidly with a small change in angle. This can lead to a situation where the speed of Unit 2 is seriously over-estimated.
Shallow Angle Collisions

Consider the following collision:

<table>
<thead>
<tr>
<th>Data Table:</th>
<th>Vehicle 1</th>
<th>Vehicle 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>3625 lb</td>
<td>4320 lb</td>
</tr>
<tr>
<td>Approach Speed</td>
<td>$S_1$</td>
<td>$S_2$</td>
</tr>
<tr>
<td>Approach Angle</td>
<td>$\alpha = 0^\circ$</td>
<td>$\psi = 355^\circ$</td>
</tr>
<tr>
<td></td>
<td>$\sin \alpha = 0$</td>
<td>$\sin \psi = -0.087$</td>
</tr>
<tr>
<td></td>
<td>$\cos \alpha = 1$</td>
<td>$\cos \psi = 0.996$</td>
</tr>
<tr>
<td>Departure Speed</td>
<td>18.49 mph</td>
<td>21.35 mph</td>
</tr>
<tr>
<td>Departure Angle</td>
<td>$\theta = 170^\circ$</td>
<td>$\phi = 167^\circ$</td>
</tr>
<tr>
<td></td>
<td>$\sin \theta = 0.173$</td>
<td>$\sin \phi = 0.224$</td>
</tr>
<tr>
<td></td>
<td>$\cos \theta = -0.984$</td>
<td>$\cos \phi = -0.974$</td>
</tr>
</tbody>
</table>
If we do the speed calculations for $v_1$ and $v_2$:

$$V_2 = 85.82 \text{ mph}$$

And

$$V_1 = 58.89 \text{ mph}$$
What if the approach angle for Unit 2 is 356°?

<table>
<thead>
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<tr>
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<td>$S_2$</td>
</tr>
<tr>
<td>Approach Angle</td>
<td>$\alpha = 0^\circ$</td>
<td>$\psi = 356^\circ$</td>
</tr>
<tr>
<td></td>
<td>$\sin \alpha = 0$</td>
<td>$\sin \psi = -0.069$</td>
</tr>
<tr>
<td></td>
<td>$\cos \alpha = 1$</td>
<td>$\cos \psi = 0.997$</td>
</tr>
<tr>
<td>Departure Speed</td>
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</tr>
<tr>
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<td>$\sin \phi = 0.224$</td>
</tr>
<tr>
<td></td>
<td>$\cos \theta = -0.984$</td>
<td>$\cos \phi = -0.974$</td>
</tr>
</tbody>
</table>
Shallow Angle Collisions

• If we do the speed calculations for $v_1$ and $v_2$:

\[ V_2 = 108.21 \text{ mph} \]

And

\[ V_1 = 85.59 \text{ mph} \]
Shallow Angle Collisions

- We may see a one degree change in the approach angle yields very different impact speeds.
- Because of this sensitivity to small approach angles, a traditional planar (360 Method) COLM analysis becomes essentially unusable.
- We will recommend that approach angles of 10° or less be handled as a collinear problem.
- A better solution, with proper training, is to use either a CRASH III or Simultaneous Equations approach.
For many, if not most, of the collisions we work on a day to day basis, the basic planar COLM solution works well.

There are cases where either significant external, impulsive forces exist or the COLM solution becomes too sensitive for us to measure angles or masses accurately enough to yield a single, reasonable solution.

Ask yourself, “Is this a closed momentum system, and can I measure what I need to the accuracy I require?”
Occupant Kinematics

*How do the people or objects inside move with respect to the vehicle?*
Dynamics

- Dynamics is the study of forces and motion with respect to the ground.
- In essence, this means we have a coordinate system fixed in space that neither translates (moves) or rotates.
- This is also referred to as an *inertial* reference frame.
Kinematics

- *Kinematics* is the study of the motion of one body with respect to another.
- In essence, this means we have coordinate system that is not fixed in space and can also rotate.
- A vehicle-fixed coordinate system is an example of this.
- This is also referred to as a *non-inertial* reference frame.
- We use kinematics to see how one vehicle may move with respect to another...or how occupants move inside a crash vehicle.
The vehicle and the people are both moving with respect to the ground.

The force on the vehicle knocks the vehicle out from under the people, who keep on moving according to Newton’s First Law.

What this means is the people, or anything else in the vehicle not attached to the vehicle, tends to move toward the applied force if our frame of reference is the vehicle itself.
An Interesting Speed Computation Problem:

A bus pulls out from a stop sign and into the path of an oncoming small car. The car runs into the side of the bus and penetrates back to the C-pillar, killing both occupants. The bus drives to the side of the road with the car still stuck underneath it. Both the mass ratio and controlled final rest position of both vehicles preclude the use of a COILM solution, and the damage to the small car is an override configuration, precluding the use of a CRASH III analysis. You can tie scratches on the hood of the car to its initial penetration under the bus. Develop an analysis for determining the impact speed of the car if you can quantify the speed of the bus. The collision is NOT at right angles.

Solution: Draw a velocity vector diagram:

Math Solution: We know angle B from the scratches on the hood of the small car. We know angle R from the two approach velocity vectors. Thus, we also know angle C = (B+R) - 180.

Use the Law of Sines:

\[
\frac{V}{\sin(B)} = \frac{V_c}{\sin(C)}
\]

Solve for \( V_c \):

\[
V_c = \frac{V \cdot \sin(C)}{\sin(B)}
\]
Determine PDOF angles.

\[ \alpha_2 = -32^0 \]

\[ \alpha_1 = 58^0 \]
In this crash, both occupants will move toward the PDOF.

For vehicle one, an unrestrained driver will leave evidence near the rear-view mirror / windshield midline.

The vehicle one passenger will leave evidence near or on the passenger side A-pillar.

Vehicle two driver will move toward the A-pillar on his side.

Vehicle two passenger will leave evidence around the rear-view mirror / windshield midline.
At IPTM Special Problems 2007, we crashed a city transit bus into a stopped 1991 Chevrolet Caprice. The impact speed was on the order of 40 mph. There were two volunteer passengers on the bus as well as the driver. Let’s look at some video...
At the Pennsylvania State Police Reconstruction Seminar in October 2007, we crashed two cars together at a high bullet vehicle speed. The impact speed was on the order of 69 mph. There were two “Rescue Andy” dummies in each vehicle. Let’s look at some video from the crash and from inside the bullet vehicle...
• From the video we have just seen, notice how the people inside tend to move toward the impact force.
• This is not what is really happening...the vehicle is being knocked out from under the people.
• With respect to the ground, the people continue moving in a straight line until they interact with the vehicle.
• Thus, the appearance of “moving toward the force”.
• In the case of the bus crash, the speed change of the bus was low, and there was little occupant motion.
The End – of the Beginning!