An Introduction To Rotational Mechanics

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Introduction

- In investigating collisions, situations are encountered where one or more of the vehicles involved experience some degree of spin or rotation.

- The rotational motion may be present along with translational motion.

- Proper interpretation and analysis of the rotational motion may present a challenge for investigators.
Objectives

- The concepts of rotational mechanics will be presented.
- Application will be discussed.
- Case examples will be presented to exemplify the use of rotational mechanics in crash analyses.
DEFINITIONS
Definitions

- Angular Displacement
  
  - Denoted by \( \theta \) (theta) and is measured in radians.
  
  - Angular displacement is analogous to translational displacement measured in feet or meters.
Definitions (cont.)

Angular Displacement (cont.)

- Formulas:

\[
\theta = \omega_o t + \frac{1}{2} \alpha t^2
\]

Translational equivalent:

\[
d = v_o t + \frac{1}{2} at^2
\]
Angular Velocity

- Change in angular displacement with respect to a change in time.
- Denoted by $\omega$ (omega) and is measured in radians per second.
- Angular velocity is analogous to translational velocity measured in ft/sec or m/sec.
Angular Velocity (cont.)

Formulas:

\[ \omega = \frac{\Delta \theta}{\Delta t} \]

\[ \omega_f = \omega_o + \alpha t \]

\[ \omega_f^2 = \omega_o^2 + 2\alpha \theta \]

Translational equivalent:

\[ v = \frac{\Delta d}{\Delta t} \]

\[ v_f = v_o + at \]

\[ v_f^2 = v_o^2 + 2ad \]
Definitions (cont.)

- Angular Acceleration
  - Change in angular velocity with respect to a change in time.
  - Denoted by " (alpha) and is measured in radians per second$^2$.
  - Angular acceleration is analogous to translational acceleration measured in ft/sec$^2$ or m/sec$^2$. 
Definitions (cont.)

- **Angular Acceleration (cont.)**
  - Formula:

\[ \alpha = \frac{\Delta \omega}{\Delta t} \]

Translational equivalent:

\[ \alpha = \frac{\Delta v}{\Delta t} \]
Definitions (cont.)

- **Moment of Inertia**
  - A measure of the resistance a body has to angular acceleration.
  - A function of both the mass and shape of the body.
  - Denoted by \( I \) (uppercase i) and is measured in slug-ft\(^2\) or lb-ft-sec\(^2\) in the English system and kg-m\(^2\) in the metric system.
  - Moment of inertia is analogous to mass, which is a measure of a body’s resistance to translational acceleration.
Definitions (cont.)

- Moment of Inertia (cont.)
  - Formula:

\[
I = \sum Mr^2
\]
Definitions (cont.)

- **Radius of Gyration**
  - A single radius which replaces all the radii of each individual particle of an object and results in the same moment of inertia.

  - Defined by the letter $k$. 
Definitions (cont.)

- Moment of Inertia using Radius of Gyration
  - Formula:

\[ I = Mk^2 \]

or

\[ k^2 = \frac{I}{M} \]
Definitions (cont.)

- **Torque**
  - Torque is produced by a force multiplied by the length of the lever arm, or moment arm, against which the force is acting.
  - Denoted by $\tau$ (tau) and is measured in ft-lbs or N-m.
  - Torque is analogous to force.
Definitions (cont.)

- Torque (cont.)
  - Formula:

\[ \tau = F \times L \]
Definitions (cont.)

- Newton’s 2nd Law for Rotation
  - Torque = Moment of Inertia X Angular Acceleration

\[ \tau = I \alpha \]

Translational equivalent:

\[ F = Ma \]
Definitions (cont.)

- Angular Impulse
  - Impulse = Torque X Time

\[ I = \tau \Delta t \]

Translational equivalent:

\[ I = F \Delta t \]
Definitions (cont.)

- Work done by Rotation
  - \( \text{Work} = \text{Torque} \times \text{Angular Displacement} \)
  - Measured in ft-lbs or joules.
Definitions (cont.)

- Work done by Rotation (cont.)
  - Formula:

\[ W = \tau \theta \]

Translational equivalent:

\[ W = Fd \]
Definitions (cont.)

- Kinetic Energy due to Rotation
  - Energy due to the rotational motion of the object.
  - Measured in ft-lbs or joules.
Kinetic Energy due to Rotation (cont.)

- Formula:

\[
Ke = \frac{1}{2} I \omega^2
\]

Translational equivalent:

\[
Ke = \frac{1}{2} Mv^2
\]
Definitions (cont.)

- **Work-Energy Theorem**
  
  - The work performed in moving an object is equal to the change in kinetic energy of the object.
  
  - Measured in ft-lbs or joules.
Definitions (cont.)

- Work-Energy Theorem (cont.)
  - Formula:

\[ \tau \theta = \frac{1}{2} I \omega^2 \]

Translational equivalent:

\[ Fd = \frac{1}{2} Mv^2 \]
Definitions (cont.)

Parallel Axis Theorem

• The relationship between the rotational inertia of a body about any axis and its rotational inertia about a different, parallel axis, separate from the first.
Definitions (cont.)

- **Parallel Axis Theorem** (cont.)
  - Formula:
  
  \[
  I = I_G + Mh^2
  \]

  Where
  - \(I_G\) = known moment of inertia
  - \(M\) = total mass of the object
  - \(h\) = perpendicular distance between the axes
Case Study #1
Case Study #1

- A bus drove off a level surface and struck the embankment on the other side.
- The bus did not totally leave the surface, the rear axles remained on the surface.
- The bus was wedged against the other side. This was first touch as well as final position.
Case Study #1

- Issue
  - What was the speed of the bus as it left the surface?
**Case Study #1**

- **Data**
  - $w = 29,465$ lbs  
  - $W.B. = 285$ in.  
  - $CM = 194.9$ in.  
  - $2 = 3^\circ = 0.052$ rad  
  - $d = 21.83$ ft.  
  - $l_p = l_G = 167,750$ slug-ft$^2$  

  - Weight of bus  
  - Wheelbase  
  - Behind the front axle  
  - Angular displacement  
  - Distance front axle traveled after leaving surface  
  - Pitch moment of inertia
Case Study #2
Case Study #2

- A truck drove off a level surface, went airborne, and landed on a lower surface.
- The truck traveled 31 feet horizontally and fell 6 feet while airborne.
- The calculated speed at take-off was 34 mph.
Case Study #2

- Opposing expert stated the truck did not go airborne but merely drove down the 10° slope to the lower surface.

- Issue
  - At what speed will the truck go airborne?
Case Study #2

Data

- $d = 31$ ft. \hspace{1cm} Horizontal airborne distance
- $h = 6$ ft. \hspace{1cm} Height fallen while airborne
- $w = 4294$ lbs \hspace{1cm} Weight of pick-up truck
- $W.B. = 11$ ft. \hspace{1cm} Wheelbase
- $CM = 6.49$ ft. \hspace{1cm} In front of the rear axle
- $\theta = 10^\circ = 0.17$ rad \hspace{1cm} Angular displacement
- $I_p = I_G = 3102.06$ lb-ft-sec$^2$ \hspace{1cm} Pitch moment of inertia
Case Study #3
Case Study #3

- A truck tractor-flatbed semitrailer was parked.
- A car, weighing 4000 lbs., struck the semitrailer in the left side.
- The force of the collision was strong enough to move the semitrailer and cause it to pivot about the kingpin.
Case Study #3

Issue

• What was the speed of the car at impact?
Case Study #3

Data

- $w_1 = 36,000$ lbs  Weight of flatbed semitrailer
- $w_2 = 4000$ lbs  Weight of car
- $w_R = 12,000$ lbs  Weight on rear tandems
- $d_1 = 18$ ft.  Distance from kingpin to CM
- $d_2 = 31$ ft.  Distance from kingpin to impact of car
- $d_3 = 36.83$ ft.  Distance from kingpin to center of tandems
- $\theta = 0.3$ rad  Angular displacement of rear of semitrailer
- $I_y = I_G = 170,000$ lb-ft-sec$^2$  Yaw moment of inertia
- $\mu = .60$  Coefficient of friction
Further Considerations
Further Considerations

How are moments of inertia of vehicles determined?

• Tables and documentation

• Calculations
Determining Moments of Inertia

The easiest method for obtaining the moments of inertia of cars and light trucks is to use a published database.

Computer programs such as “Expert AutoStats®” provide data for the pitch, yaw and roll moments of inertia for a wide range of vehicles.
Determining Moments of Inertia (cont.)

- The moments of inertia can also be determined through the use of published regression equations.
- This article describes the physical measurement methodology used to determine inertial parameters for cars and light trucks.
- The following regression equations were developed and presented in the article.
Regression Equations for Cars

- Yaw Moment = (1.03 x Weight) – 1206
- Pitch Moment = (0.99 x Weight) – 1149
- Roll Moment = (0.18 x Weight) – 150
Regression Equations for Light Trucks

- Yaw Moment = $(1.03 \times \text{Weight}) - 1343$
- Pitch Moment = $(1.12 \times \text{Weight}) - 1657$
- Roll Moment = $(0.22 \times \text{Weight}) - 235$
Further Considerations (cont.)

Determining Moments of Inertia (cont.)

- Commercial Vehicle Moments of Inertia
  - The study was conducted by the University of Michigan Transportation Research Institute.
Further Considerations (cont.)

Determining Moments of Inertia (cont.)

- Commercial Vehicle Moments of Inertia (cont)
  - Moments of Inertia will help us determine how commercial vehicles may react in certain driving conditions.
  - Method for calculating various moments of inertia are found in the reference.
  - Yaw and Pitch Moments are very similar for commercial vehicles.
  - All references are to Moments of Inertia about an axis going through the CM.
Further Considerations (cont.)

Determining Moments of Inertia (cont.)

◆ Commercial Vehicle Moments of Inertia (cont)

• The **Yaw Moment** will assist in the study of response times, rearward amplification, transient braking, and response to disturbances such as wind loading.

• The **Pitch Moment** is useful in studying transient braking response, such as ABS or driver initiated intermittent braking. The pitch moment may also assist us with certain speed calculations.

• The **Roll Moment** is smaller than the other two. It is used when studying dynamic rollover situations.
Further Considerations (cont.)

Determining Moments of Inertia (cont.)

- Truck & Tractor Yaw and Pitch Moments
  - Ford 9000 Tractor: 26,560 lb-ft-sec^2
  - GMC Astro 95 Tractor: 20,123 lb-ft-sec^2
  - Ford 800 Tractor: 15,946 lb-ft-sec^2
  - IH Tractor: 14,730 lb-ft-sec^2
  - GMC Astro 95 Dump, empty: 14,713 lb-ft-sec^2
  - GMC Tractor: 11,546 lb-ft-sec^2
  - GMC 6500 V-8 Dump, empty: 10,970 lb-ft-sec^2
Further Considerations (cont.)

Determining Moments of Inertia (cont.)

- Empty Semitrailer Pitch and Yaw Moments
  - 48’, Tandem, 40’ WB, 13,800 lb: 110,739 lb-ft-sec²
  - 45’, Tandem, 37’ WB, 13,043 lb: 91,157 lb-ft-sec²
  - 42’, Tandem, 36’ WB, 12,286 lb: 78,751 lb-ft-sec²
  - 28’, Single, 22.8’ WB, 6,753 lb: 39,627 lb-ft-sec²
  - 27’, Single, 21’ WB, 6,500 lb: 34,600 lb-ft-sec²
Further Considerations (cont.)

Determining Moments of Inertia (cont.)

- **Loaded Semitrailer Pitch and Yaw Moments**
  - Homogeneous cargo @ 14 lb/ft³
    - 48’, Tandem, 40’ WB, 60,500 lb: 403,539 lb-ft·sec²
    - 45’, Tandem, 37’ WB, 56,843 lb: 333,878 lb-ft·sec²
    - 42’, Tandem, 36’ WB, 53,086 lb: 278,217 lb-ft·sec²
    - 28’, Single, 22.8’ WB, 33,952 lb: 100,921 lb-ft·sec²
    - 27’, Single, 21’ WB, 32,750 lb: 89,848 lb-ft·sec²
Further Considerations

How are eccentric collisions, where one vehicle is “swept” aside, handled?

- Utilize the “Effective Mass Ratio”
- Calculate $V$
- Perform traditional conservation of linear momentum computation.
Further Considerations (cont.)

Utilizing Effective Mass Ratio (cont.)

- Consider the collision where a stationary vehicle is eccentrically struck in the side, towards the front, by another vehicle.
- The collision pivots the stationary vehicle out of the way as the bullet vehicle continues to a stop.
- The stationary vehicle does not have a great deal of post collision translational motion.
Further Considerations (cont.)

Utilizing Effective Mass Ratio (cont.)

- In this collision the target vehicle was not entirely displaced by the collision.
- The full effect of the mass of the target vehicle was not felt by the bullet vehicle.
- A fair representation of the departure speed of the target vehicle might not be obtained by traditional methods.
- Therefore, the effective mass ratio of each vehicle should be calculated and used to determine delta-Vs.
- Then a traditional COLM calculation can be performed.
Utilizing Effective Mass Ratio (cont.)

Effective Mass Ratio

- Formula:

\[ \gamma = \frac{k^2}{h^2 + k^2} \]

Where \( k \) = radius of gyration
\( h \) = lever arm on which the collision force acts
Use the calculated effective mass ratios of both vehicles to calculate the delta-V of the target vehicle.
Utilizing Effective Mass Ratio (cont.)

- Delta-V formula:

\[ \Delta v_2 = \sqrt{\frac{2g\gamma_2 (E_1 + E_2)}{w_2 \left( 1 + \frac{\gamma_2 w_2}{\gamma_1 w_1} \right)}} \]

Where

- \( \gamma_1, \gamma_2 \) = respective effective mass ratios of bullet and target vehicles
- \( w_1, w_2 \) = respective weights of bullet and target vehicles
- \( E_1, E_2 \) = respective amount of energy needed to damage the bullet and target vehicles
- \( g \) = gravity
Further Considerations (cont.)

Utilizing Effective Mass Ratio (cont.)

- Having calculated delta-V, this is also the departure speed of the target vehicle since it was stationary.
- The impact speed of the bullet vehicle can now be calculated using the traditional COLM equation.

\[ v_1 = v_3 + \frac{w_2 v_4}{w_1} - \frac{w_2 v_2}{w_1} \]
CONCLUSIONS
Conclusions

- A large number of situations can be solved using standard rectilinear techniques and equations.
- However there are those situations which present rotational issues which must be considered.
- Understanding and utilizing proper theories and formulas from rotational mechanics allows solutions to these situations to be found.
- As reconstructionists, we must continually strive to add to our toolbox of knowledge to be able to do the best job we can.